Weak entire solution with finite excited intervals for a reaction-interface system

Yan-Yu Chen (NTU)

Joint work with H. Ninomiya and C.-H. Wu

2021 TMS Annual Meeting

Outline

- 1. Introduction
- 2. Main Results
- 3. One and two excited intervals cases
- 4. Non-existence for more than three intervals

1. Introduction

FitzHugh-Nagumo type equations

(FHN)
$$\begin{cases} u_t = \Delta u + \frac{1}{\varepsilon^2}(f_\varepsilon(u) - \varepsilon\beta v), & x \in \mathbb{R}^n,\ t \in \mathbb{R}, \\ v_t = g(u,v), & x \in \mathbb{R}^n,\ t \in \mathbb{R}, \end{cases}$$

where $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ and

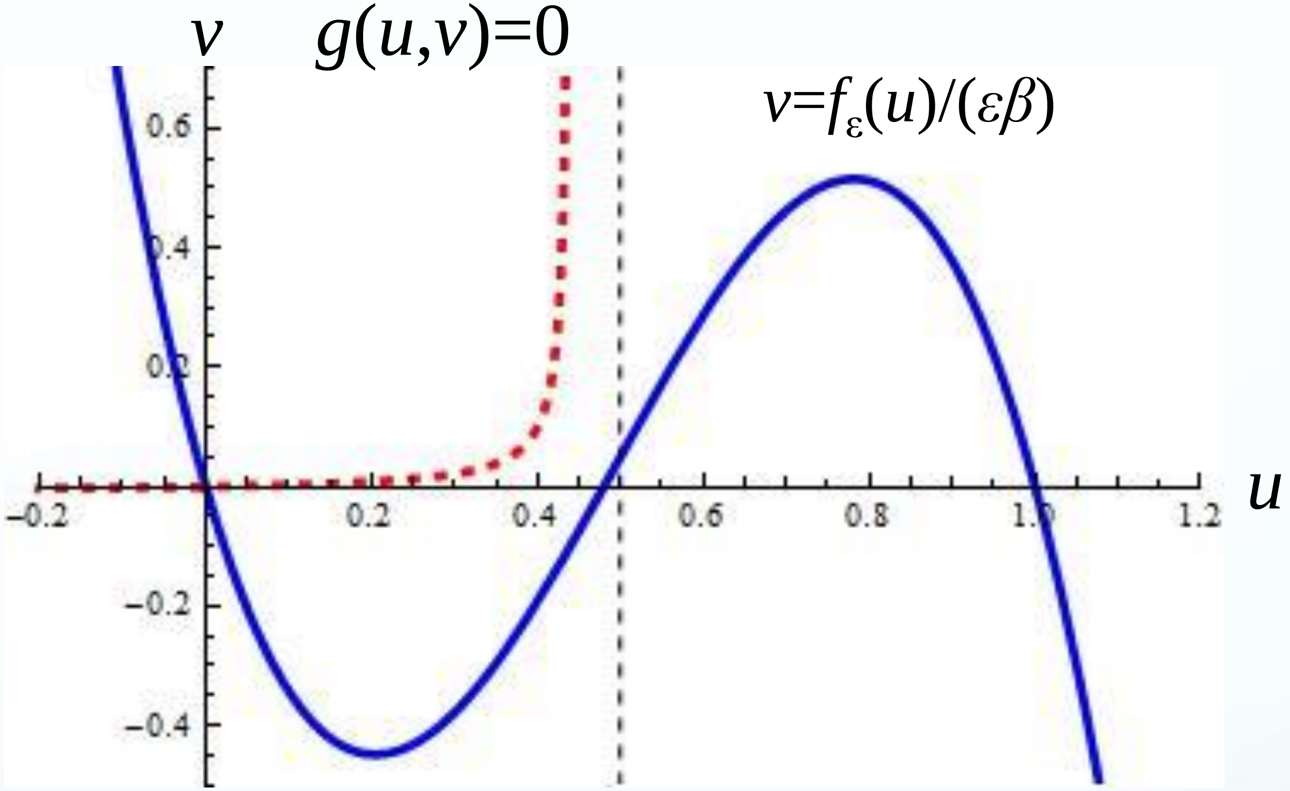
$$f_\varepsilon(u) := u(1-u)\left(u - \frac{1}{2} + \varepsilon\alpha\right), \quad g(u,v) = g_1 u - \frac{g_2 v}{g_4 + g_3 v}$$

with positive constants $g_1, \cdots, g_4$ and α.

u and v represent the activator and inhibitor in the excitable media, respetively.

Here, we assume that

$$\frac{g_2}{g_3} < \frac{g_1}{2}$$



$$\varepsilon = 0.01, \alpha = 2, \beta = 10, g_1 = 1, g_2 = 0.04, g_3 = 1, g_4 = 0.01$$

Singular limit problem

Chen-Kohsaka-Ninomiya 2014

By a singular limit analysis (taking $\varepsilon \to 0$), (FHN) is reduce to

$$\text{(SLP-n)} \quad \begin{cases} V = W(v) - (n-1)\kappa, & x \in \partial\Omega(t), t \in \mathbb{R}, \\ v_t = g(\chi_{\Omega(t)}, v), & x \in \mathbb{R}^n, t \in \mathbb{R} \end{cases}$$

Ω(t): the excited region ($u^\varepsilon \to 1$ as $\varepsilon \to 0$)

V : Outer normal velocity of the interface

κ : Mean curvature of the interface

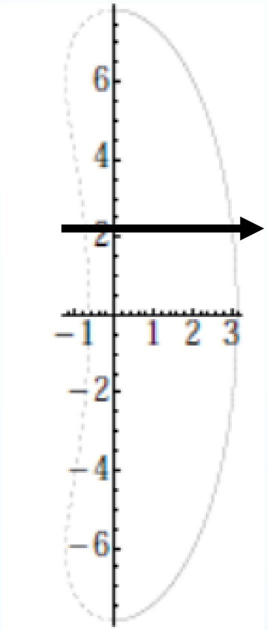
$\chi_{\Omega(t)}$: Characteristic function of Ω(t)

$$W(v) = a - bv, \qquad a = \sqrt{2}\alpha, \qquad b = 6\sqrt{2}\beta$$

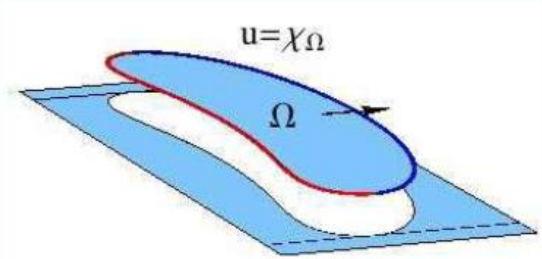
Previous works for singular limit of the reaction-diffusion equations:
Alfaro-Hilhorst-Matano, X. Y. Chen, X. Chen, Fife,
Fife-Hsiao, Giga-Goto-Ishii, Hilhorst-Nishiura-Mimura

Results for multi-dimensional traveling spot

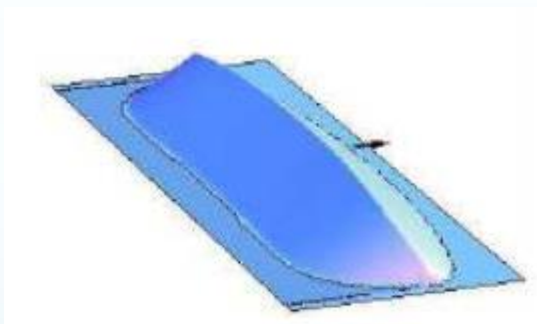
CKN, 2014: For c<a



2D traveling spot

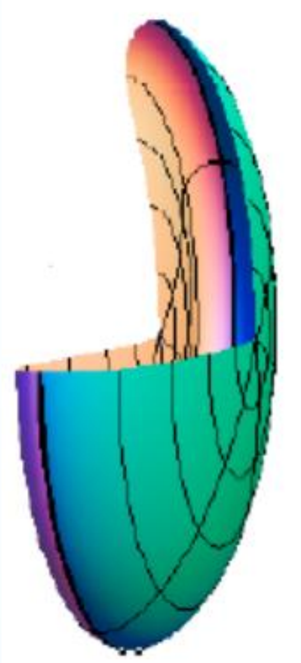


Profile of *u*



Profile of *v*

CNT, 2015



3D traveling spot

Planar wave exist for c=a

Ninomiya and Wu 2017: V-shaped traveling curve exists for c>a

In [1, 2], we consider the following initial value problem with suitable initial data:

$$\text{(SLP-1)} \quad \begin{cases} V = W(v), & x \in \partial\Omega(t),\ t > 0, \\ v_t = g(\mathbf{1}_{\Omega(t)}, v), & x \in \mathbb{R},\ t > 0, \\ \Omega(0) = \Omega_0 := \bigcup_{j=1}^{m} (x_{2j-1}^0, x_{2j}^0), \quad v(x,0) = v_0(x), & x \in \mathbb{R}, \end{cases}$$

where the initial data satisfies

(H1) (Boundedness) Ω_0 consists of m disjoint bounded intervals and $v_0 \geq 0$ is a bounded Lipschitz function defined in $\mathbb{R}$ with

$$M := \|v_0\|_{L^\infty(\mathbb{R})}.$$

(H2) (Well-posedness) $W(v_0(x)) \neq 0$ for any $x \in \partial\Omega(0)$.

[1] Y.-Y. Chen, H. Ninomiya, C.-H. Wu, Global Existence and Uniqueness of Solutions for One-dimensional Reaction-interface Systems, submitted, https://arxiv.org/abs/2104.04971.
[2] Y.-Y. Chen, H. Ninomiya, C.-H. Wu, Global dynamics on one-dimensional excitable media, SIAM J. Math. Anal., 53-6 (2021), pp. 7081—7112

In [1], we give the definition of classical solution. Due to the annihilation, we introduce the weak solution such that the solution can be defined globally in time. We show the existence and uniqueness of the weak solution.

In [2], we obtain asymptotic behavior satisfies the following trichotomy:

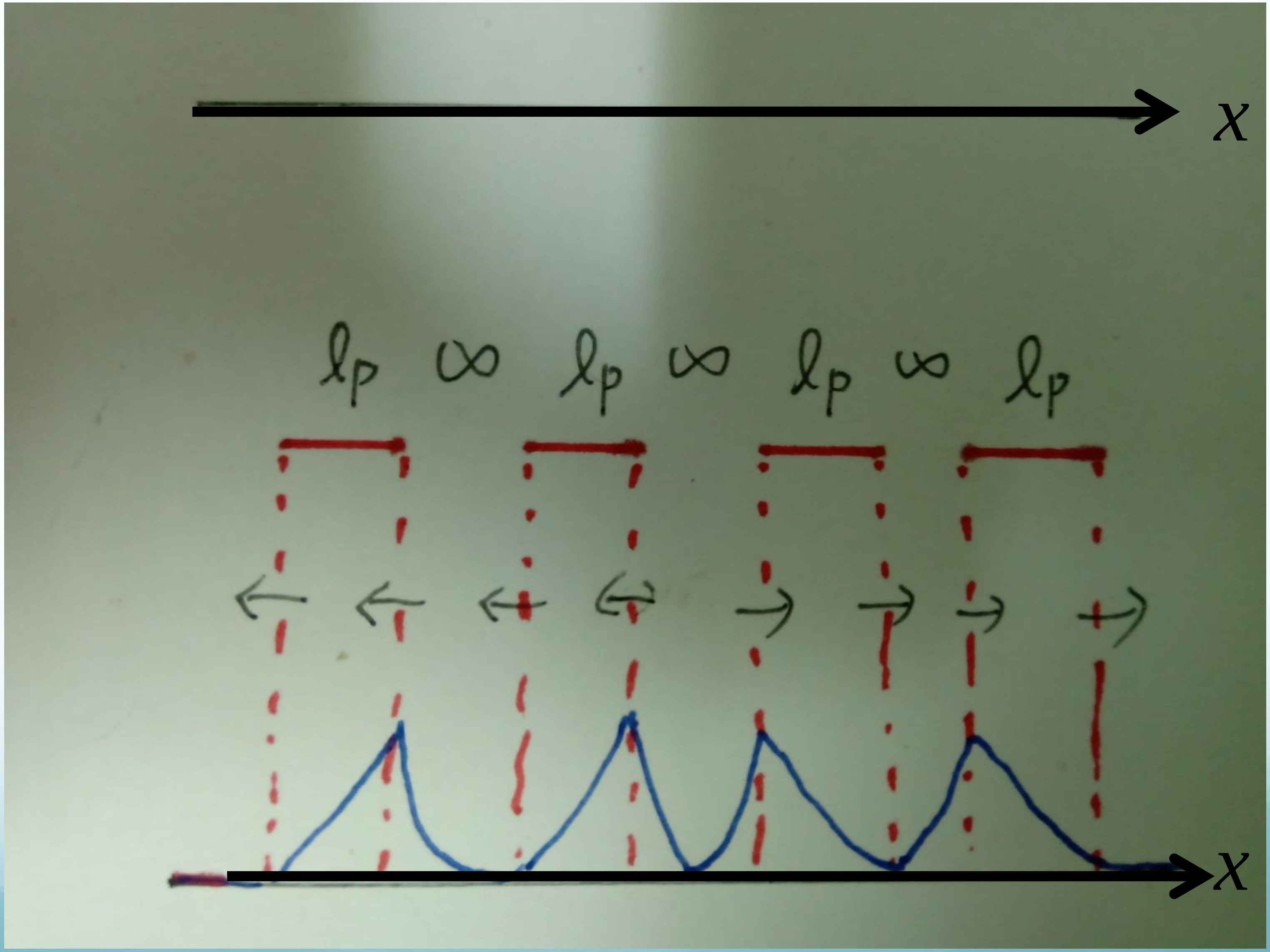
(i) the solution of (SLP-1) converges to the resting state 0 in $L^\infty(R)$;

(ii) the solution of (SLP-1) converges to a series of traveling pulses propagating in either the same direction or both directions;

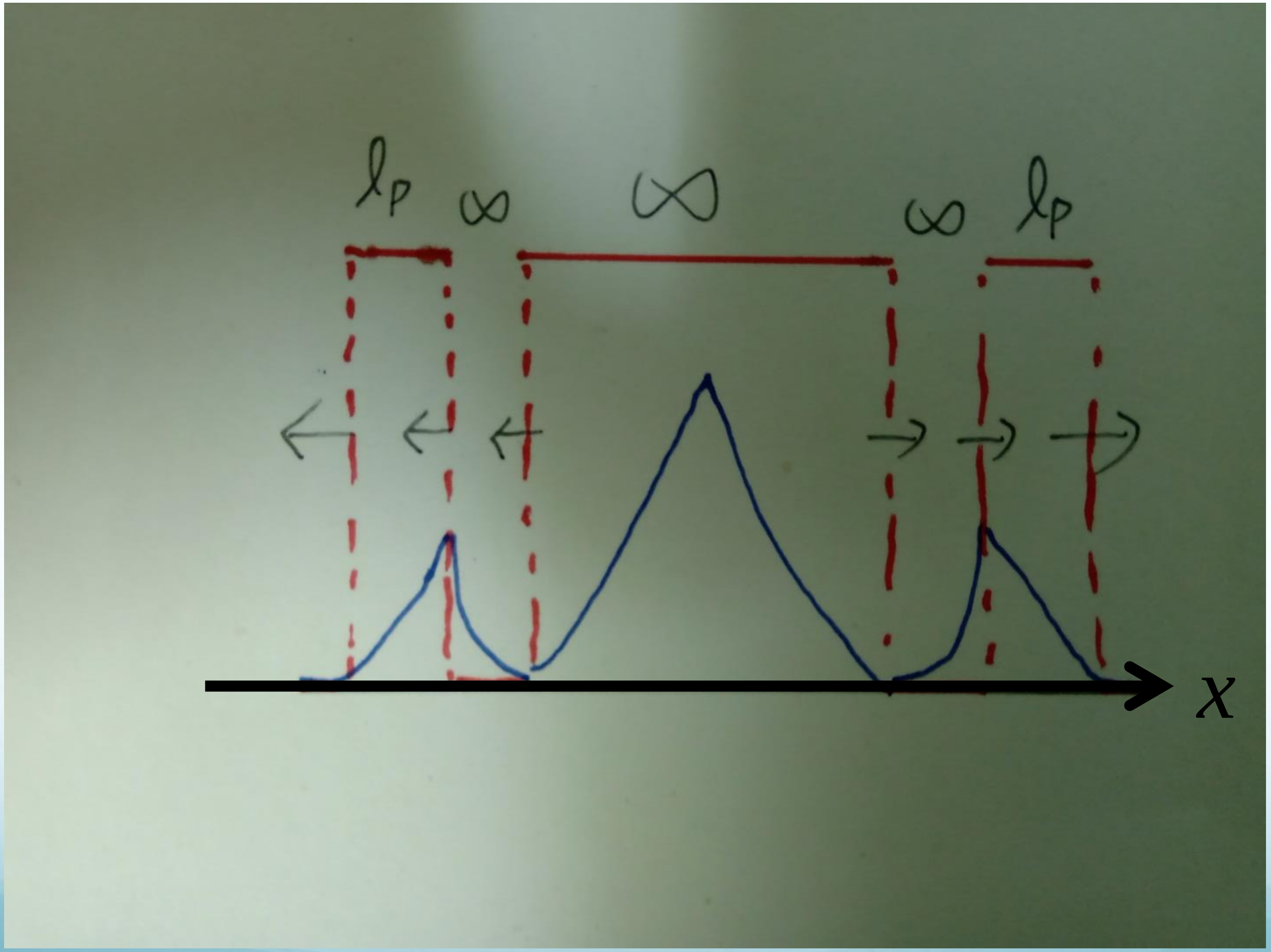
(iii) the solution of (SLP-1) converges to a propagating wave consisting of multiple traveling pulses and two traveling fronts propagating in both directions.

Type (i)

Type (ii)



Type(iii)



Motivation

In this work, we consider the following reaction-interface problem:

$$\text{(RI)} \quad \begin{cases} V = W(v(x,t)), & x \in \partial\Omega(t),\ t \in \mathbb{R}, \\ v_t = g(\mathbf{1}_{\Omega(t)}, v), & x \in \mathbb{R},\ t \in \mathbb{R}. \end{cases}$$

To obtain a more complete description of the global dynamics for (RI), we study its entire solutions which are meant by the solutions defined globally in time and space. We also expect there are more interesting behaviors as time tends to minus infinity.

2. Main Results

Special solution in one-dimension (n=1)

Definition

A traveling pulse solution with speed c means a solution (Ω, v) of (RI) takes the following form:

$$\Omega(t) = \Omega_P(t) := \{x \mid ct - \ell_P < x < ct\}, \quad v(x,t) = \varphi_P(x - ct)$$

for some positive constant ℓ_P, and a function $\varphi_P(\cdot) \in C(\mathrm{R}) \cap \mathrm{C}^1(\mathrm{R} \setminus \{0, -\ell_\mathrm{P}\})$ satisfying $\varphi_P(\pm\infty) = 0$.

Special solution in one-dimension (n=1)

Theorem [2, Theorem 2.1]

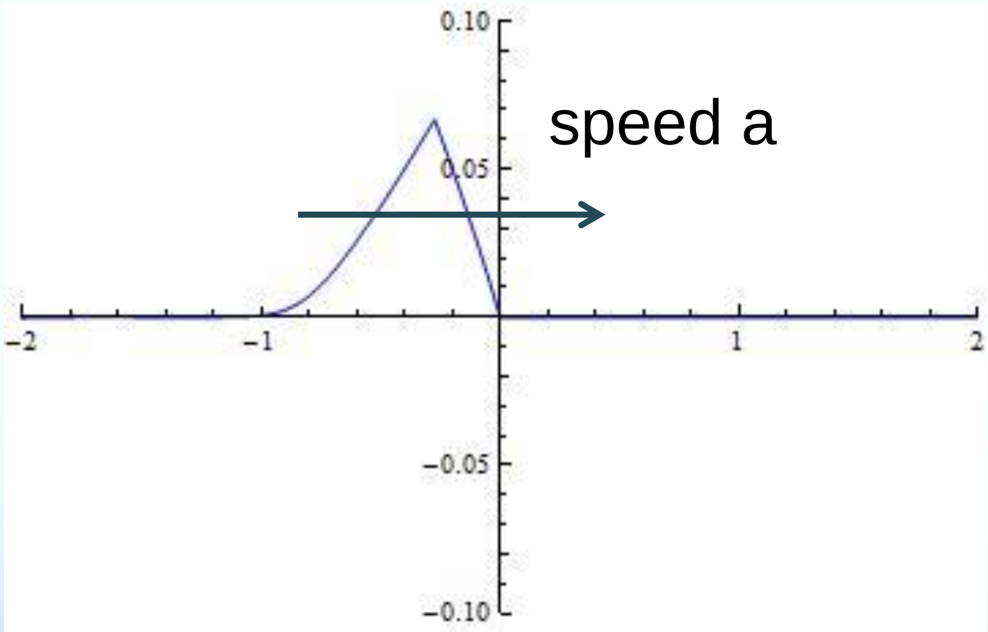
There exists a traveling pulse solution $(\Omega_P(t), \varphi_P(x-at))$ of (RI) with speed a, where

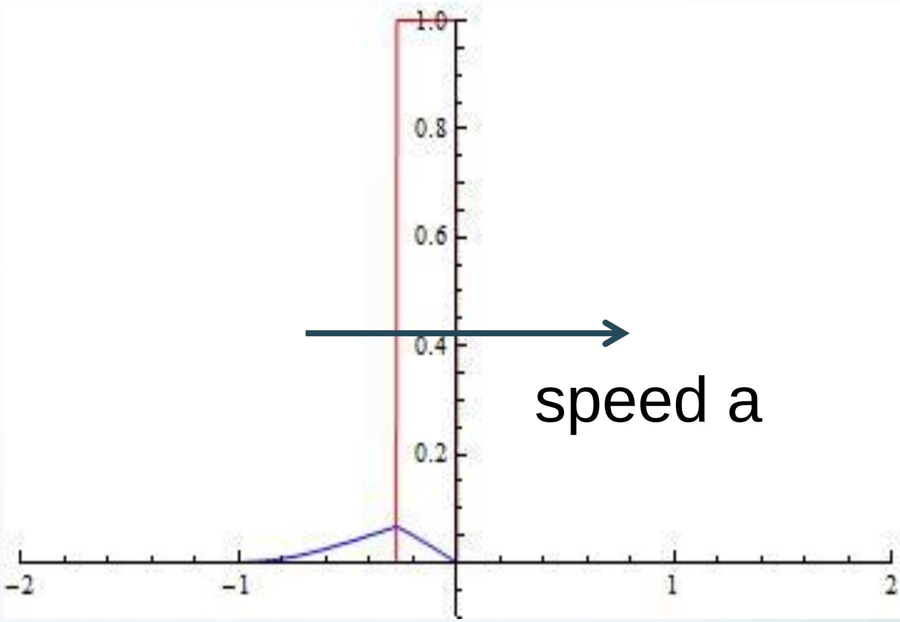
$$\Omega_P(t) := \{x \mid at - \ell_P < x < at\}, \quad \ell_P := a\, G_1^{-1}\left(\frac{2a}{b}\right),$$

$$\varphi_P(z) := \begin{cases} G_1\left(\dfrac{(-z)^+}{a}\right), & \text{if } -\ell_P \le z, \\ G_0\left(-\dfrac{z+\ell_P}{a} + G_0^{-1}\left(\dfrac{2a}{b}\right)\right), & \text{if } z \le -\ell_P. \end{cases}$$

$$G_0^{-1}(v) := \int_M^v \frac{d\xi}{g(0,\xi)}, \quad G_1^{-1}(v) := \int_0^v \frac{d\xi}{g(1,\xi)},$$

Traveling pulse





Red: the profile of u; Blue: the profile of v

Ancient solution and classical entire solution

Definition (i) (Ω, v) *is called an ancient solution of* (RI) *with* m *excited intervals for* $t < T$, *if there exist* $x_k \in C^1((-\infty, T))$, $k \in \{1, 2, \cdots, 2m\}$, *a constant* M *and*

$$v \in C(\mathbb{R} \times (-\infty, T)) \cap C^1\Big(\mathbb{R} \times (-\infty, T) \setminus \{x = x_k(t),\ t < T,\ k \in \mathcal{M}\}\Big)$$

such that

$$\begin{aligned}
&0 \le v(x,t) \le M \quad \text{for all } (x,t) \in \mathbb{R} \times (-\infty, T),\\
&-\infty < x_i(t) < x_{i+1}(t) < \infty \quad \text{for all } t < T,\ i = 1, \ldots, 2m-1,\\
&W(v(x_k(t), t)) \neq 0 \quad \text{for all } t < T,\ k \in \mathcal{M},\\
&\Omega := \bigcup_{-\infty < t < T} [\Omega(t) \times \{t\}] = \bigcup_{-\infty < t < T} \left[\bigcup_{j=1}^{m} (x_{2j-1}(t), x_{2j}(t)) \times \{t\}\right]
\end{aligned}$$

and the following equations hold pointwisely:

$$\begin{aligned}
x_k'(t) &= (-1)^k W\Big(v(x_k(t), t)\Big) := (-1)^k [a - bv(x_k(t), t)],\ t < T,\ k \in \mathcal{M},\\
v_t &= g(\mathbf{1}_{\Omega(t)}, v) \quad \text{in } \mathbb{R} \times (-\infty, T).
\end{aligned}$$

(ii) *If* $T = \infty$, *we call the solution* (Ω, v) *a classical entire solution of* (RI) *with* m *excited intervals.*

Weak entire solution

Motivated by [1], we define the weak entire solution of (RI) as follows.

$$X := \left\{ (\Omega, v) \,\middle|\, \begin{array}{l} v \in BC(\mathbb{R}^2) \text{ and is globally Lipschitz in } x, \text{ unifromly in } t, \\ v \geq 0 \text{ in } \mathbb{R}^2,\ \Omega \subset \mathbb{R}^2 \text{ is open, } \partial\Omega \text{ is Lipschitz} \\ W(v) \neq 0 \text{ on } \overline{\bigcup_{t\in\mathbb{R}} \partial\Omega(t) \times \{t\}}. \end{array} \right\}$$

$$\Omega(t) := \mathrm{int}_{\mathbb{R}}\{x \in \mathbb{R} |\, (x,t) \in \overline{\Omega}\},$$

Definition We say that a pair (Ω, v) is *a weak entire solution* of (RI) for $t \in \mathbb{R}$, if

(i) $(\Omega, v) \in X$ and

$$0 = \int_{-\infty}^{\infty} \int_{\Omega(t)} \varphi_t dxdt + \int_{\partial\Omega\cap(\mathbb{R}^2)} W(v)\varphi|n_1|d\sigma,$$

$$0 = \int_{-\infty}^{\infty} \int_{\mathbb{R}} \Big(v\psi_t + g(\mathbf{1}_{\Omega(t)}, v)\psi \Big) dxdt$$

for any $\varphi \in H^1(\mathbb{R}; L^2(\mathbb{R}))$ and $\psi \in H^1(\mathbb{R}; L^2(\mathbb{R}))$ such that φ and ψ have a compact support in $\mathbb{R}^2$;

(ii) if $B(x_0, r_0) \times \{t_0\} \subset \Omega$ (resp. $\subset \Omega^c$) for some $r_0 > 0$ and $t_0 \in \mathbb{R}$, then there exists a positive constant τ_0 depending only on r_0 such that

$$\{x_0\} \times [t_0, t_0 + \tau_0] \subset \Omega \text{ (resp. } \subset \Omega^c);$$

Weak entire solution with *m* excited intervals

$$\mathcal{X} := \left\{ (D, w) \;\middle|\; w \in C(\mathbb{R}),\; D \text{ is open in } \mathbb{R} \text{ and } W(w)|_{\partial D} \neq 0 \right\}$$

D is the union of a countable collection of disjoint open intervals.

$Z[(D, w)] := \inf\{k \in \mathbb{N} \cup \{\infty\} \mid \overline{D} = \cup_{i=1}^{k} \overline{J}_i$ for some open disjoint intervals $J_1, \cdots, J_k$ $\}$.

$Z[(\Omega(t), v(\cdot, t))]$ can be seen as the number of the excited intervals at time t.

Definition We say that a pair (Ω, v) is a *weak entire solution* of (RI) with m excited intervals if it is a weak solution of (RI) for $t \in \mathbb{R}$ and $\limsup_{t \to -\infty} Z[(\Omega(t), v(\cdot, t))] = m < \infty$.

Main Theorem

Theorem 1

If (Ω, v) is a weak entire solution of (RI) with m excited intervals, then there exists a time $T \leq \infty$ such that it is an ancient solution with m excited intervals for $t < T$. Conversely, if (Ω, v) is an ancient solution for $t < T$ with m excited intervals , then it can be extended uniquely to be a weak entire solution with m excited intervals.

Theorem 2 (one interval case)

Any weak entire solution of (RI) with one excited interval must be a traveling pulse solution

Theorem 3 (two intervals case)

Any weak entire solution of (RI) with two excited intervals must be a two-facing traveling pulse solution annihilating at some time T_E (up to shift), namely,

$$\Omega_E(t) := (at - \ell_P - \xi_1, at - \xi_1) \cup (-at + \xi_2, -at + \ell_P + \xi_2),$$
$$v_E(x,t) := \varphi_P(x - at + \xi_1) + \varphi_P(-at - x + \xi_2)$$

for $t < T_E := (\xi_1 + \xi_2)/(2a)$ with some constants ξ_1 and ξ_2.

Theorem 4 (Non-existence)

There are no weak entire solutions of (RI) with m excited intervals when $3 \leq m < \infty$.

3. One and two excited intervals cases

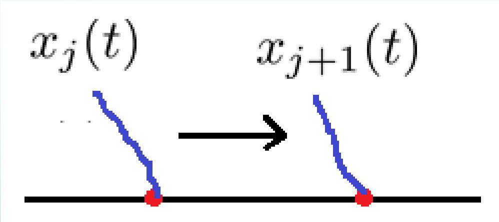
Properties of ancient solution

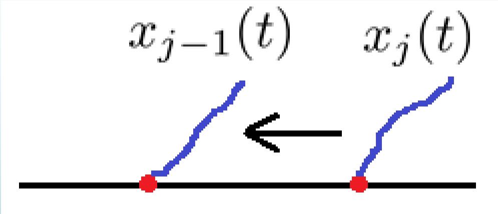
Lemma 1

Assume that (Ω, v) is an ancient solution for $t < T \leq \infty$.
If there exists j such that $x'_j(t) < 0$ for $t < T \leq \infty$,
then $x'_{j+1}(t) < 0$ for $t < T \leq \infty$.

Lemma 2

Assume that (Ω, v) is an ancient solution for $t < T \leq \infty$.
If there exists j such that $x'_j(t) > 0$ for $t < T \leq \infty$,
then $x'_{j-1}(t) > 0$ for $t < T \leq \infty$.



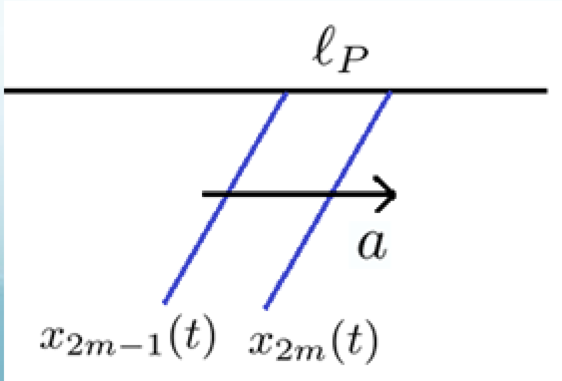


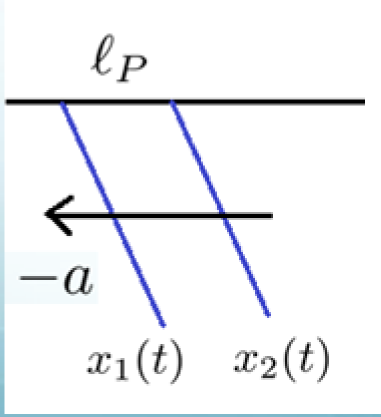
Lemma 3 (right-end excited interval)

Assume that (Ω, v) is an ancient solution for $t < T \leq \infty$ with m excited interval and $x'_{2m}(t) > 0$ for any $t < T$. Then $v(x,t) \equiv 0$ for $x \geq x_{2m}(t)$ and $t < T$. Moreover, $x'_{2m}(t) \equiv a$ and $x_{2m}(t) - x_{2m-1}(t) \equiv \ell_p$ for all $t < T$.

Lemma 4 (left-end excited interval)

Assume that (Ω, v) is an ancient solution for $t < T \leq \infty$ with m excited interval and $x'_1(t) < 0$ for any $t < T$. Then $v(x,t) \equiv 0$ for $x \leq x_1(t)$ and $t < T$. Moreover, $x'_1(t) \equiv -a$ and $x_2(t) - x_1(t) \equiv \ell_p$ for all $t < T$.





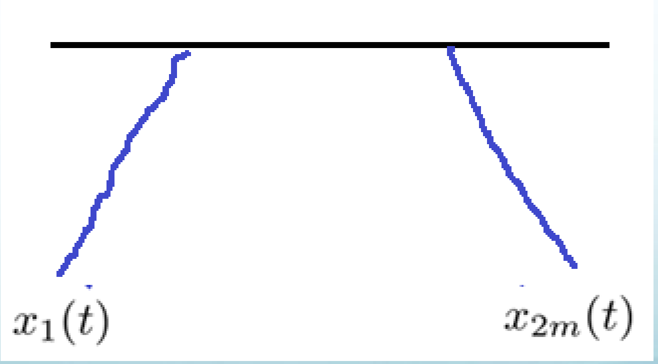
Properties of ancient solution with more than two intervals

Lemma 5 (right-end interface)

There is no ancient solution (Ω, v) with m excited intervals satisfying $x'_{2m}(t) > 0$ if $2 \leq m < \infty$

Lemma 6 (left-end interface)

There is no ancient solution (Ω, v) with m excited intervals satisfying $x'_1(t) < 0$.



Sketch proof of Theorem 2:

By Theorem 1, we assume that $\Omega(t) = (x_1(t), x_2(t))$, and either $x_2'(t) > 0$ or $x_2'(t) < 0$ for $t < T$ for some $-\infty < T \leq \infty$

Case1. $x_2'(t) > 0$:

By Lemma 3, we have $x_2(t) = at$ and $x_1 = at - \ell_P$.

For $x \geq a\,t$, we have $v(x,t) = 0 = \varphi_P(x - at)$.

For $a\,t - \ell_P \leq x \leq a\,t$, we have

$$v(x,t) = G_1\left(\frac{a\,t - x}{a}\right) = \varphi_P(x - at).$$

For $x \leq a\,t - \ell_P$, we have

$$v(x,t) = G_0\left(\frac{a\,t - \ell_P - x}{a} + G_0^{-1}\left(\frac{2a}{b}\right)\right) = \varphi_P(x - at).$$

Hence, the solution is exactly the traveling pulse moving rightwards for $t < T$.

Consider the system (RI) for $t \geq T - \varepsilon$ for some small $\varepsilon > 0$. By the result of the existence and uniqueness of the solution shown in [1], the solution with initial data $(\Omega(T-\varepsilon), v(\cdot, T-\varepsilon))$ is still the traveling pulse moving rightwards for $t > T - \varepsilon$. Hence, Theorem 2 holds when $x_2'(t) > 0$ for $t < T$.

Case 2. $x_2'(t) < 0$:

First, we would like to show that $x_1'(t) < 0$.
For contradiction, we assume that $x_1'(t) > 0$ for $t < T$. Then we can pick (x_0, t_0) such that $x_1(t_0) < x_0 < x_2(t_0)$ and $t_0 < T$, and $v_t(x_0, t)/g(1, v) = 1$ for all $t < t_0$. By integrating over (s, t_0), we have

$$G_1^{-1}(v(x_0, s)) = G_1^{-1}(v(x_0, t_0)) - t_0 + s.$$

Taking $s \to -\infty$, we see that $v(x_0, s) \to -\infty$ due to $G_1^{-1}(\infty) = \infty$, which contradicts the boundedness of v.

By Lemma 4, we have $x_1(t) = -at$ and $x_2 = -at + \ell_P$.

Use the same arguement as Case 1, we can show that the solution is the traveling pulse moving leftwards.

Sketch proof of Theorem 3:

First, we show that he solution (Ω_E, v_E) is an ancient solution for $t < T_E := (\xi_1 + \xi_2)/(2a)$.
Set $x_1(t) = at - \ell_P - \xi_1$, $x_2(t) = at - \xi_1$, $x_3(t) = -at + \xi_2$ and $x_4(t) = -at + \ell_P + \xi_2$. Also,

$$v_E(x,t) = \begin{cases} \varphi_P(x - at + \xi_1), & \text{if } x \le at - \xi_1, \\ 0, & \text{if } at - \xi_1 \le x \le -at + \xi_2, \\ \varphi_P(-at - x + \xi_2), & \text{if } x \ge -at + \xi_2. \end{cases}$$

Then, by the definition of φ_P, $v_E(x,t)$ is bounded and satisfies $(v_E)_t = g(\mathbf{1}_{\Omega_E(t)}, v_E)$.

$$\begin{aligned} x_1'(t) &= a = (-1)(a - b(2a/b)) = (-1)(a - bv_E(x_1(t),t)), \\ x_2'(t) &= a = (-1)^2(a - b(0)) = (-1)^2(a - bv_E(x_2(t),t)), \\ x_3'(t) &= -a = (-1)^3(a - b(0)) = (-1)^3(a - bv_E(x_3(t),t)), \\ x_4'(t) &= -a = (-1)^4(a - b(2a/b)) = (-1)^4(a - bv_E(x_4(t),t)). \end{aligned}$$

Therefore, (Ω_E, v_E) is an ancient solution.

By Theorem 1, we assume $\Omega(t) = (x_1(t), x_2(t)) \cup (x_3(t), x_4(t))$.

From Lemma 5-6, we have $x_1'(t) > 0$ and $x_4'(t) < 0$. By using the same argument as in the proof of Theorem 2, we have $x_2'(t) > 0$ and $x_3'(t) < 0$.

By using the same argument as in the proof of Lemma 3-4, we obtain the $v(x,t) \equiv 0$ for $x_2(t) \leq x \leq x_3(t)$ and $t < T$. Then we have $x_2'(t) \equiv a$, $x_3'(t) \equiv -a$ and $x_2(t) - x_1(t)$ $= x_4(t) - x_3(t) \equiv \ell_p$ for $t < T$.

Together with the argument used in the proof of Theorem 2, we see that the ancient solution of (RI) with two excited intervals must be a two-facing traveling pulse solution (Ω_E, v_E) with some translation.

Sketch proof of Theorem 4:

By Theorem 1, we assume $\Omega(t) = \bigcup_{j=1}^{m}(x_{2j-1}(t), x_{2j}(t))$ where

$$-\infty < x_1(t) < \cdots < x_{2j-1}(t) < x_{2j}(t) < \cdots < x_{2m}(t) < \infty.$$

By Lemma 5-6, we have $x_1'(t) > 0$ and $x_{2m}'(t) < 0$ for $t < T$. By the similar argument as in the proof of Theorem 2, we have $x_2'(t) > 0$ and $x_{2m-1}'(t) < 0$ for $t < T$.

By using the contradiction argument, we show there is a unique positive integer $k \in \{1, \ldots, m-1\}$ such that $x_{2k}'(t) > 0$ and $x_{2k+1}'(t) < 0$ for $t < T$.

By the proof of Lemma 3, it follows that

$$v(x,t) = 0 \quad \text{for } x_{2k}(t) \le x \le x_{2k+1}(t) \text{ and } t < T.$$

Thanks to the uniqueness of k, it is easy to see that $x_i'(t) > 0$ for $i = 1, \cdots, 2k$ and $x_i'(t) < 0$ for $i = 2k+1, \cdots, 2m$. By using the parallel proof of Lemma 5 to get a contradiction. This completes the proof.

Conclusion of today's talk:

(1) Show the existence of the weak entire solution with one excited interval and two excited intervals

(2) Prove the non-existence of the weak entire solution with more than three excited intervals.

Some future works.

(1) How about infinitely many intervals?

(2) Capture the pattern on multi-dimension by using (SLP-n).

Thank you for your attention